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On the Behaviour of Zeros of Jacobi Polynomials

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Denote by $x_{n,k}(\alpha,\beta)$ and $x_{n,k}(\lambda)=x_{n,k}(\lambda-1/2,\lambda-1/2)$ the zeros, in decreasing order, of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$ and of the ultraspherical (Gegenbauer) polynomial $C_n^{\lambda}(x)$, respectively. The monotonicity of $x_{n,k}(\alpha,\beta)$ as functions of α and β, $\alpha,\beta>-1$, is investigated. Necessary conditions such that the zeros of $P_n^{(a,b)}(x)$ are smaller (greater) than the zeros of $P_n^{(\alpha,\beta)}(x)$ are provided. A. Markov proved that $x_{n,k}(a,b)<x_{n,k}(\alpha,\beta)$ ($x_{n,k}(a,b)>x_{n,k}(\alpha,\beta)$) for every $n\in\mathbb{N}$ and each k, $1\leqslant k\leqslant n$ if $a>\alpha$ and $b<\beta$ ($a<\alpha$ and $b>\beta$). We prove the converse statement of Markov's theorem. The question of how large the function $f_n(\lambda)$ could be such that the products $f_n(\lambda)\,x_{n,k}(\lambda)$, $k=1,\ldots,[n/2]$ are increasing functions of λ, for $\lambda>-1/2$, is also discussed. Elbert and Siafarikas proved that $f_n(\lambda)=(\lambda+(2n^2+1)/(4n+2))^{1/2}$ obeys this property. We establish the sharpness of their result.

1. INTRODUCTION AND STATEMENT OF RESULTS

The behaviour of the zeros $x_{n,k}(\alpha,\beta)$, $k=1,\ldots,n$, of the Jacobi polynomial $P_n^{(\alpha,\beta)}(x)$, arranged in decreasing order, as functions of the parameters α and β, $\alpha,\beta>-1$, has been of permanent interest because of their nice electrostatic

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interpretation and of their important role as nodes of Gaussian quadrature formulae, just to mention a few. In particular, the monotonicity properties of the positive zeros $x_{n,k}(\lambda)$, $k=1, \ldots, [n/2]$, of the ultraspherical polynomial $C_n^{\lambda}(x)$, for $\lambda > -1/2$, have been under intensive investigation. In order to do this, powerful analytic and discrete techniques which provide sufficient conditions for monotonicity have been developed. Among others we mention Sturm's theorem [18, Theorem 1.82.1] and its integral version [5], A. Markov's theorem [15; 18, Theorem 6.12.1] and the Hellmann–Feynman theorem [9, 10, 12]. However, these methods hardly indicate to what extent the corresponding results are sharp.

We are interested in two important questions.

1. Given a pair (α, β), $\alpha, \beta > -1$, find all (a, b), $a, b > -1$, for which the zeros of $P_n^{(a,b)}(x)$ are smaller (greater) than the zeros of $P_n^{(\alpha,\beta)}(x)$. In other words, the problem is to determine the pairs (a, b) such that the inequalities $x_{n,k}(a, b) < x_{n,k}(\alpha, \beta)$ (or $x_{n,k}(a, b) > x_{n,k}(\alpha, \beta)$) hold for every positive integer n and for any k, $1 \leqslant k \leqslant n$. A result of A. Markov [15; 18, Theorem 6.21.1] states that the zeros of $P_n^{(\alpha,\beta)}(x)$ are decreasing functions of α and increasing functions of β. Equivalently, the inequalities $x_{n,k}(a, b) < x_{n,k}(\alpha, \beta)$, $n=2, 3, \ldots$, $k=1, \ldots, n$, hold for $a > \alpha$, $b < \beta$, and $x_{n,k}(a, b) > x_{n,k}(\alpha, \beta)$, $n=2, 3, \ldots$, $k=1, \ldots, n$, for $a < \alpha$, $b > \beta$. To the best of our knowledge nothing was known about the mutual location of $x_{n,k}(a, b)$ and $x_{n,k}(\alpha, \beta)$ when (a, b) belongs to the sectors $\{a > \alpha, b > \beta\}$ and $\{a < \alpha, b < \beta\}$ and this is the first question we are interested in.

In the recent paper [4], we proved the inequalities $x_{n,1}(a, b) < x_{n,1}(\alpha, \beta)$ for the largest zeros of $P_n^{(a,b)}(x)$ and $P_n^{(\alpha,\beta)}(x)$ in the following cases

$$\beta > \alpha,\ a > \alpha,\ b - \beta < a - \alpha,$$

and

$$\beta < \alpha,\ a > \alpha,\ \alpha + \beta > 0,\ (\alpha+1)(b-\beta) < (\beta+1)(a-\alpha),\ (a, b) \notin \varDelta,$$

where $\varDelta = \varDelta(\alpha, \beta)$ is the triangle with vertices at (α, β), $(2\alpha - \beta, \beta)$, and $(2\alpha+1, 2\beta+1)$.

2. Stieltjes [17] proved that the positive zeros of $C_n^{\lambda}(x)$ decrease when λ increases. The problem of finding the extremal function $f_n(\lambda)$ which forces the products $f_n(\lambda)\, x_{n,k}(\lambda)$, $k=1, \ldots, [n/2]$, to increase has been discussed in [1, 3, 6, 8, 13]. Recently Elbert and Siafarikas [6] proved that $[\lambda + (2n^2+1)/(4n+2)]^{1/2}\, x_{n,k}(\lambda)$, $k=1, \ldots, [n/2]$, are increasing functions of λ, for $\lambda > -1/2$, thus extending a result of Ahmed, Muldoon, and Spigler [1] and proving a conjecture of Ismail, Letessier, and Askey [10, 11].

It is interesting to know to what extent the multiplier function $f_n(\lambda) = (\lambda + (2n^2+1)/(4n+2))^{1/2}$ is close to the extremal one. This is the second question we are interested in. There are strong indications that $f_n(\lambda)$ should be of the form $f_n(\lambda) = (\lambda + c_n)^{1/2}$, where c_n depends only on n. We refer to the introduction of [6] for some of the arguments in support of this statement. On the other hand, as it was pointed out in [3], the best possible function $f_n(\lambda)$ that forces $f_n(\lambda)\, x_{n,k}(\lambda)$, $k = 1, ..., [n/2]$, to increase is the one for which $f'_n(\lambda)/f_n(\lambda)$ is minimal. Thus $f_n(\lambda) = (\lambda + c_n)^{1/2}$ is the extremal function for this problem if c_n is the largest possible.

In this paper we employ the classical Routh–Hurwitz stability criterion in order to answer the above two questions. Our results read as follows.

THEOREM 1.1. *Let (α, β), $\alpha, \beta > -1$, be a pair of parameters and n be a positive integer.*

If $x_{n,k}(a, b) < x_{n,k}(\alpha, \beta)$ for every k, $k = 1, ..., n$, then

$$b < \min\left\{\beta + \frac{n+\beta}{n+\alpha}(a-\alpha),\ \beta + \frac{n+\beta}{1+\alpha}(a-\alpha)\right\}. \tag{1}$$

if $x_{n,k}(a, b) > x_{n,k}(\alpha, \beta)$ for every k, $k = 1, ..., n$, then

$$b > \max\left\{\beta + \frac{n+\beta}{n+\alpha}(a-\alpha),\ \beta + \frac{n+\beta}{1+\alpha}(a-\alpha)\right\}. \tag{2}$$

Let us fix the point (α, β) and consider the regions in the (a, b)-plane where (1) and (2) hold. Inequality (1) holds when (a, b) belongs to the sector below the lines l_1 and l_2, where l_1 passes through (α, β) and $(-n, -n)$, and l_2 passes through (α, β) and $(-1, -n)$. Inequality (2) holds when (a, b) is in the sector above these lines.

THEOREM 1.2. *Let (α, β), $\alpha, \beta > -1$, be a fixed pair of parameters. Then the inequalities $x_{n,k}(a, b) < x_{n,k}(\alpha, \beta)$ hold for every positive integer n and for each k, $k = 1, ..., n$, if and only if*

$$a > \alpha \qquad \text{and} \qquad b < \beta. \tag{3}$$

The inequalities $x_{n,k}(a, b) > x_{n,k}(\alpha, \beta)$ hold for every $n \in \mathbb{N}$ and for each k, $k = 1, ..., n$, if and only if

$$a < \alpha \qquad \text{and} \qquad b > \beta. \tag{4}$$

Markov's theorem asserts that *all* the zeros of *all* the polynomials $P_n^{(a,b)}(x)$ precede the corresponding zeros of $P_n^{(\alpha,\beta)}(x)$ if the vector $(a, b) - (\alpha, \beta)$ is in the fourth quadrant and the opposite inequalities $x_{n,k}(a, b) > x_{n,k}(\alpha, \beta)$ hold for all admissible n and k if the vector $(a, b)-(\alpha, \beta)$ is in the second quadrant. In this paper we prove that the inequalities (3) are *necessary* conditions in order that *all* the zeros of *all* the polynomials $P_n^{(a,b)}(x)$ precede the corresponding zeros of $P_n^{(\alpha,\beta)}(x)$ and, similarly, (a, b) necessarily must belong to the sector (4) provided $x_{n,k}(a, b) > x_{n,k}(\alpha, \beta)$ for *every* $n \in \mathbb{N}$ and *each* k, $1 \leqslant k \leqslant n$.

Our contribution to the second problems is as follows:

THEOREM 1.3. *Let n be a positive integer. If $f_n(\lambda)$ is positive and $f_n(\lambda)\, x_{n,k}(\lambda)$, $k=1, ..., [n/2]$, are increasing functions of λ for $\lambda > -1/2$, then*

$$\frac{f'_{2n}(\lambda)}{f_{2n}(\lambda)} > \frac{1}{2(n+\lambda)} \tag{5}$$

and

$$\frac{f'_{2n+1}(\lambda)}{f_{2n+1}(\lambda)} > \frac{1}{2(n+\lambda+1)}. \tag{6}$$

Moreover, if $(\lambda+c_n)^{1/2}\, x_{n,k}(\lambda)$, $k=1, ..., [n/2]$, are increasing functions of λ for $\lambda > -1/2$, then

$$c_{2n} < \frac{4n^2+n+1}{4n+2} \tag{7}$$

and

$$c_{2n+1} < \frac{4n^2+7n+9}{4n+6}. \tag{8}$$

Inequalities (5) and (6) provide lower bounds for the logarithmic derivative of the extremal multiplier function for the problem under discussion.

Obviously (7) and (8) are equivalent to

$$c_{2n} < n - \frac{1}{4} + \frac{3}{2(4n+2)}$$

and

$$c_{2n+1} < n + \frac{1}{4} + \frac{15}{4(2n+3)}.$$

On the other hand, the above mentioned result of Elbert and Siafarikas states that the products $f_n(\lambda)\, x_{n,k}(\lambda)$, $k=1, ..., [n/2]$, increase with λ if

$$f_n(\lambda) = (\lambda + (2n^2+1)/(4n+2))^{1/2} \tag{9}$$

and this can be rewritten as

$$f_{2n}^2(\lambda) = \lambda + n - \frac{1}{4} + \frac{3}{4(4n+1)}$$

and

$$f_{2n+1}^2(\lambda) = \lambda + n + \frac{1}{4} + \frac{3}{4(4n+3)}.$$

Thus Theorem 1.3 shows that the multiplier (9) is asymptotically sharp.

2. THE ROUTH–HURWITZ THEOREM AND MONOTONICITY OF ZEROS OF ORTHOGONAL POLYNOMIALS

Let $\{p_n(x;\tau)\}$ be a sequence of parametric orthogonal polynomials. We shall apply the stability criterion of Routh–Hurwitz to obtain necessary and sufficient conditions for monotonicity of the zeros of $p_n(x;\tau)$ as functions of the parameter τ. We refer to Gantmacher [7, Chap. 15] and Marden [14, Chap. 9] for comprehensive information on the stability theory. Here we only provide the necessary definitions and formulate the Hurwitz theorem. With every polynomial

$$f(z) = f_n z^n + f_{n-1} z^{n-1} + f_{n-2} z^{n-2} + f_{n-3} z^{n-3} + \cdots, \qquad f_n \neq 0,$$

we associate a Hurwitz matrix which is formed as follows. Set $f_{-1} = f_{-2} = \cdots = 0$ and construct the two line block

$$\begin{pmatrix} f_{n-1} & f_{n-3} & \cdots \\ f_n & f_{n-2} & \cdots \end{pmatrix},$$

where the first line contains f_{n-2k-1}, $k=0, 1, ...,$ and the second line is composed by the coefficients f_{n-2k}, $k=0, 1, ...,$ of $f(z)$. Then the Hurwitz matrix $H(f)$ of $f(z)$ is composed by the above block in its first two lines, the next two lines of $H(f)$ contain the same block shifted one position to the right, the fifth and the sixth lines contain this block shifted two positions to the right, and so forth. Thus

$$H(f) = \begin{pmatrix} f_{n-1} & f_{n-3} & f_{n-5} & \cdots & 0 \\ f_n & f_{n-2} & f_{n-4} & \cdots & 0 \\ 0 & f_{n-1} & f_{n-3} & \cdots & 0 \\ 0 & f_n & f_{n-2} & \cdots & 0 \\ . & . & . & \cdots & . \end{pmatrix}.$$

The polynomial $f(z) = f_n z^n + f_{n-1} z^{n-1} + \cdots + f_0$ with real coefficients f_j and with positive leading coefficient f_n is called Hurwitz or stable if all its zeros have negative real parts. The following is the celebrated Hurwitz theorem which is sometimes called the Routh–Hurwitz criterion.

THEOREM 2.A. *The polynomial $f(z)$ is stable if and only if the first n principal minors of the corresponding Hurwitz matrix $H(f)$ are positive.*

Another classical result we need is the Theorem of Hermite–Biehler (see Obrechkoff [16]) which reads as follows.

THEOREM 2.B. *The real polynomials $U(z)$ and $V(z)$, whose degrees are equal or differ by one, have only real and interlacing zeros if and only if all the zeros of the polynomial*

$$F(z) = U(z) + iV(z)$$

belong to one and the same half-plane with respect to the real line.

As an immediate consequence of the latter two theorems we obtain a necessary and sufficient condition such that the zeros of two real polynomials are real, negative, and interlace. In order to facilitate the formulation of the result and our further discussion we shall say that the polynomials $h(z)$ and $g(z)$ of degree m form a positive pair if their leading coefficients are positive and their zeros $x_1, ..., x_m$ and $y_1, ..., y_m$ are distinct, real, negative, and interlace in the following way:

$$y_m < x_m < y_{m-1} < x_{m-1} < \cdots < y_1 < x_1.$$

We shall succinctly denote the latter by $\bar{y} \prec \bar{x}$.

THEOREM 2.C. *The polynomial $f(z) = h(z^2) + zg(z^2)$ is a Hurwitz polynomial if and only if $h(z)$ and $g(z)$ form a positive pair.*

This result appears as Theorem 13 in [7, p. 228].

Consider the sequence $\{p_n(x;\tau)\}$ of parametric polynomials which are orthogonal on the interval $x \in (c, d)$ when $\tau \in (p, q)$ and whose coefficients are continuous functions of τ. Suppose the leading coefficients of $p_n(x;\tau)$ are positive. We shall denote by $\zeta_k(\tau)$,

$$c < \zeta_n(\tau) < \zeta_{n-1}(\tau) < \cdots < \zeta_1(\tau) < d,$$

the zeros of $p_n(x;\tau)$ arranged in decreasing order. Let

$$p_n(x;\tau) = a_0(\tau) + a_1(\tau)(x-d) + \cdots + a_n(\tau)(x-d)^n, \qquad a_n(\tau) > 0, \quad (10)$$

be the Taylor expansion of $p_n(x;\tau)$ at d. Since the zeros $\zeta_k(\tau)$, $k = 1, ..., n$ of $p_n(x;\tau)$ are distinct and belong to (c, d), then all the coefficients $a_j(\tau)$, $j = 0, ..., n$, are positive. Let $q_n(x;\tau)$ be the polynomial

$$q_n(x;\tau) = a_0(\tau) + a_1(\tau)\, x + \cdots + a_n(\tau)\, x^n,$$

and

$$\tilde{q}_n(x;\tau) = a_0(\tau)\, x^n + \cdots + a_1(\tau)\, x + a_n(\tau)$$

be its reciprocal. Denote by $H(p_n;\tau_1,\tau_2)$ the Hurwitz matrix associated with the polynomial

$$f_{2n+1}(x;\tau_1,\tau_2) := q_n(x^2;\tau_1) + x q_n(x^2;\tau_2).$$

We have

$$H(p_n;\tau_1,\tau_2) = \begin{pmatrix} a_n(\tau_1) & a_{n-1}(\tau_1) & a_{n-2}(\tau_1) & \cdots & 0 \\ a_n(\tau_2) & a_{n-1}(\tau_2) & a_{n-2}(\tau_2) & \cdots & 0 \\ 0 & a_n(\tau_1) & a_{n-1}(\tau_1) & \cdots & 0 \\ 0 & a_n(\tau_2) & a_{n-1}(\tau_2) & \cdots & 0 \\ . & . & . & \cdots & . \end{pmatrix}.$$

Similarly, $\tilde{H}(p_n;\tau_1,\tau_2)$ denotes the Hurwitz matrix associated with

$$f^*_{2n+1}(x;\tau_1,\tau_2) := \tilde{q}_n(x^2;\tau_1) + x\tilde{q}_n(x^2;\tau_2).$$

Thus

$$\tilde{H}(p_n;\tau_1,\tau_2) = \begin{pmatrix} a_0(\tau_1) & a_1(\tau_1) & a_2(\tau_1) & \cdots & 0 \\ a_0(\tau_2) & a_1(\tau_2) & a_2(\tau_2) & \cdots & 0 \\ 0 & a_0(\tau_1) & a_1(\tau_1) & \cdots & 0 \\ 0 & a_0(\tau_2) & a_1(\tau_2) & \cdots & 0 \\ . & . & . & \cdots & . \end{pmatrix}.$$

For any j, $1 \leqslant j \leqslant 2n+1$, denote by $\Delta_j(p_n; \tau_1, \tau_2)$ and $\tilde{\Delta}_j(p_n; \tau_1, \tau_2)$ the jth principal minor of $H(p_n; \tau_1, \tau_2)$ and $\tilde{H}(p_n; \tau_1, \tau_2)$, respectively. For the first few j we have

$$\Delta_1(p_n; \tau_1, \tau_2) = a_n(\tau_1), \qquad \Delta_2(p_n; \tau_1, \tau_2) = \begin{vmatrix} a_n(\tau_1) & a_{n-1}(\tau_1) \\ a_n(\tau_2) & a_{n-1}(\tau_2) \end{vmatrix},$$

$$\Delta_3(p_n; \tau_1, \tau_2) = \begin{vmatrix} a_n(\tau_1) & a_{n-1}(\tau_1) & a_{n-2}(\tau_1) \\ a_n(\tau_2) & a_{n-1}(\tau_2) & a_{n-2}(\tau_2) \\ 0 & a_n(\tau_1) & a_{n-1}(\tau_1) \end{vmatrix},$$

$$\tilde{\Delta}_1(p_n; \tau_1, \tau_2) = a_0(\tau_1), \qquad \tilde{\Delta}_2(p_n; \tau_1, \tau_2) = \begin{vmatrix} a_0(\tau_1) & a_1(\tau_1) \\ a_0(\tau_2) & a_1(\tau_2) \end{vmatrix},$$

$$\tilde{\Delta}_3(p_n; \tau_1, \tau_2) = \begin{vmatrix} a_0(\tau_1) & a_1(\tau_1) & a_2(\tau_1) \\ a_0(\tau_2) & a_1(\tau_2) & a_2(\tau_2) \\ 0 & a_0(\tau_1) & a_1(\tau_1) \end{vmatrix}.$$

Now we are ready to formulate the principal results in this section.

THEOREM 2.1. *Let the coefficients $a_k(\tau)$ in the representation* (10) *of the parametric orthogonal polynomial $p_n(x; \tau)$ be continuous functions of τ. Then*:

(i) *The inequalities*

$$\zeta_k(\tau_2) < \zeta_k(\tau_1), \qquad k = 1, \ldots, n, \tag{11}$$

hold for any τ_2 in a sufficiently small neighbourhood of τ_1 if and only if $\Delta_j(p_n, \tau_1, \tau_2) > 0$ for $j = 1, \ldots, 2n+1$;

(ii) *The inequalities*

$$\zeta_k(\tau_1) < \zeta_k(\tau_2), \qquad k = 1, \ldots, n, \tag{12}$$

hold for any τ_2 in a sufficiently small neighborhood of τ_1 if and only if $\tilde{\Delta}_j(p_n, \tau_1, \tau_2) > 0$ for $j = 1, \ldots, 2n+1$.

Proof. First we prove statement (i). Since $p_n(x; \tau)$ are orthogonal in (c, d), then the zeros of $q_n(x; \tau)$ are negative and distinct. On the other hand, these zeros are continuous functions of the parameter τ because the coefficients depend continuously on τ. Then the zeros of $q_n(x; \tau_2)$ precede the zeros of $q_n(x; \tau_1)$, where $|\tau_1 - \tau_2| < \varepsilon$ with a sufficiently small ε if and only if the polynomials $q_n(x; \tau_1)$ and $q_n(x; \tau_2)$ form a positive pair. Now the statement of the theorem follows from Theorems 2.A and 2.C.

In order to prove (ii) we need to observe that $h(z)$ and $g(z)$ form a positive pair if and only if $z^m g(1/z)$ and $z^m h(1/z)$ do.

Remark. We can formulate various results similar to Theorem 2.1. For example, sequences of orthogonal polynomials whose coefficients depend continuously on many parameters can be considered. Then τ is understood to be a vector. Theorem 2.1 can be modified also for polynomials orthogonal on a semi-infinite line or to orthogonal polynomials with respect to an even weight function on a symmetric with respect to the origin interval. This is done by a linear or quadratic transformation. In particular, in the latter case the sequence $\{p_n(x;\tau)\}$ is reduced to two sequence of orthogonal polynomials through the simple procedure described in Section 8 of the first chapter of Chihara's book [2]. We omit the details.

3. PROOFS OF THE MAIN RESULTS

Proof of Theorem 1.1. We use the representation of the Jacobi polynomial in terms of terminating hypergeometric function,

$$P_n^{(\alpha,\beta)}(x)=\frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(-n,n+\alpha+\beta+1;\alpha+1;\frac{1-x}{2}\right).$$

Here $(a)_k$ denotes the Pochhammer symbol. Then by the linear transformation $y=(x-1)/2$ we obtain the polynomial

$$q_n^{(\alpha,\beta)}(y)=\sum_{k=0}^{n}\binom{n}{k}\frac{(n+\alpha+\beta+1)_k}{(\alpha+1)_k}\,y^k$$

whose zeros are $(x_k(\alpha,\beta)-1)/2$. Similarly, the zeros of

$$q_n^{(a,b)}(y)=\sum_{k=0}^{n}\binom{n}{k}\frac{(n+a+b+1)_k}{(a+1)_k}\,y^k$$

are $(x_k(a,b)-1)/2$.

The $(2n+1)\times(2n+1)$ matrix $H_{2n+1}(\alpha,\beta;a,b)$ is defined as follows. Its only nonzero elements are

$$h_{1,j}=\binom{n}{j-1}\frac{(n+\alpha+\beta+1)_{n+1-j}}{(\alpha+1)_{n+1-j}},\qquad j=1,\dots,n,$$

$$h_{2,j}=\binom{n}{j-1}\frac{(n+a+b+1)_{n+1-j}}{(a+1)_{n+1-j}},\qquad j=1,\dots,n,$$

and

$$h_{2k+1,j+k}=h_{1,j},\qquad h_{2k+2,j+k}=h_{2,j}.$$

In other words, $H_{2n+1}(\alpha, \beta; a, b)$ is the Hurwitz matrix associated with the polynomial

$$f_{2n+1}(y) = q_n^{(\alpha, \beta)}(y^2) + y q_n^{(a, b)}(y^2).$$

Suppose (a, b) is in a sufficiently small neighborhood of (α, β). Then, by Theorem 2.1, the zeros of $P_n^{(a, b)}(x)$ precede the one of $P_n^{(\alpha, \beta)}(x)$ if and only if all the principal minors $\Delta_j(\alpha, \beta; a, b)$ of $H_{2n+1}(\alpha, \beta; a, b)$ are positive. In particular,

$$\Delta_2(\alpha, \beta; a, b) = n \frac{(n+\alpha+\beta+1)_{n-1}}{(\alpha+1)_{n-1}} \frac{(n+a+b+1)_{n-1}}{(a+1)_{n-1}} \times \left(\frac{2n+\alpha+\beta}{\alpha+n} - \frac{2n+a+b}{a+n}\right)$$

must be positive. This is equivalent to the inequality

$$b-\beta < \frac{n+\beta}{n+\alpha}(a-\alpha).$$

In other words, (a, b) must lie in the half-plane below the line which passes through the points (α, β) and $(-n, -n)$.

Observe that $\Delta_2(\alpha, \beta; a, b) = -\Delta_2(a, b; \alpha, \beta)$. Hence, if, for (a, b) close to (α, β), the zeros of $P_n^{(\alpha, \beta)}(x)$ precede the one of $P_n^{(a, b)}(x)$, then (a, b) must belong to the opposite half-plane.

Now let $\tilde{H}_{2n+1}(\alpha, \beta; a, b)$ denote the $(2n+1)\times(2n+1)$ Hurwitz matrix associated with

$$f^*_{2n+1}(y) = \tilde{q}_n^{(\alpha, \beta)}(y^2) + y\tilde{q}_n^{(a, b)}(y^2).$$

The 2×2 principal minor of $\tilde{H}_{2n+1}(\alpha, \beta; a, b)$ is

$$\tilde{\Delta}_2(\alpha, \beta, a, b) = n\left(\frac{n+a+b}{a+1} - \frac{n+\alpha+\beta}{\alpha+1}\right).$$

Then Theorem 2.1 implies that if (a, b) is sufficiently close to (α, β) and $x_{n,k}(\alpha, \beta) < x_{n,k}(a, b)$ for $k = l, ..., n$, then, $\tilde{\Delta}_2(\alpha, \beta; a, b)$ must be positive. This is equivalent to

$$b-\beta > \frac{n+\beta}{\alpha+1}(a-\alpha).$$

The latter means that (a, b) must lie in the half-plane above the line which passes through the points (α, β) and $(-1, -n)$.

The argument $\tilde{\Delta}_2(\alpha, \beta; a, b) = -\Delta_2(a, b; \alpha, \beta)$ implies that if the inequalities $x_{n,k}(a, b) < x_{n,k}(\alpha, \beta)$ hold for all the zeros of $P_n^{(a,b)}(x)$ and $P_n^{(\alpha,\beta)}(x)$ when (a, b) is close to (α, β), then (a, b) must lie in the half-plane below the line through (α, β) and $(-1, -n)$. This completes the proof of Theorem 1.1.

In order to prove Theorem 1.2 we need to formulate an immediate consequence of Theorem 1.1. Considering the intersections of the sectors (1) and (2), when n runs over the positive integers, we immediately obtain:

COROLLARY 3.1. *Let $\alpha, \beta > -1$ be any fixed parameters.*

(i) *If $\alpha > \beta$ and the inequalities $x_{n,k}(a, b) < x_{n,k}(\alpha, \beta)$ hold for every positive integer n and for each k, $k = 1, \ldots, n$, then*

$$a > \alpha \qquad \text{and} \qquad b < \beta + \frac{\beta+1}{\alpha+1}(a-\alpha). \tag{13}$$

(ii) *If $\alpha > \beta$ and the inequalities $x_{n,k}(a, b) < x_{n,k}(\alpha, \beta)$ for every $n \in \mathbb{N}$ and for each k, $1 \leqslant k \leqslant n$, then*

$$a < \alpha \qquad \text{and} \qquad b > \beta + \frac{\beta+1}{\alpha+1}(a-\alpha). \tag{14}$$

(iii) *If $\alpha < \beta$ and the inequalities $x_{n,k}(a, b) < x_{n,k}(\alpha, \beta)$ hold for every positive integer n and for each k, $k = 1, \ldots, n$, then*

$$a > \alpha \qquad \text{and} \qquad b < \beta + a - \alpha. \tag{15}$$

(iv) *If $\alpha < \beta$ and the inequalities $x_{n,k}(a, b) > x_{n,k}(\alpha, \beta)$ hold for every positive integer n and for each k, $k = 1, \ldots, n$, then*

$$a < \alpha \qquad \text{and} \qquad b > \beta + a - \alpha. \tag{16}$$

Thus, when (a, b) is outside the sectors described by (13), (14), (15), *and* (16), *then there simultaneously exist indices n, k and n', k' for which the opposite inequalities $x_{n,k}(a, b) < x_{n,k}(\alpha, \beta)$ and $x_{n',k'}(a, b) > x_{n',k'}(\alpha, \beta)$ hold.*

The regions represented by the above inequalities are shown on Fig. 1. If $\alpha > \beta$, then sector (13) is the union of the sectors which contain the sign "<" and the point "1," and sector (14) is the union of the sectors which contain the sign ">" and the point "2." Similarly, when the point (α, β) is above the line $a = b$, then the sector (15) is the union of the sectors which contain the sign "<" and the point "3," and the sector (16) is the union of the sectors containing the sign ">" and the point "4."

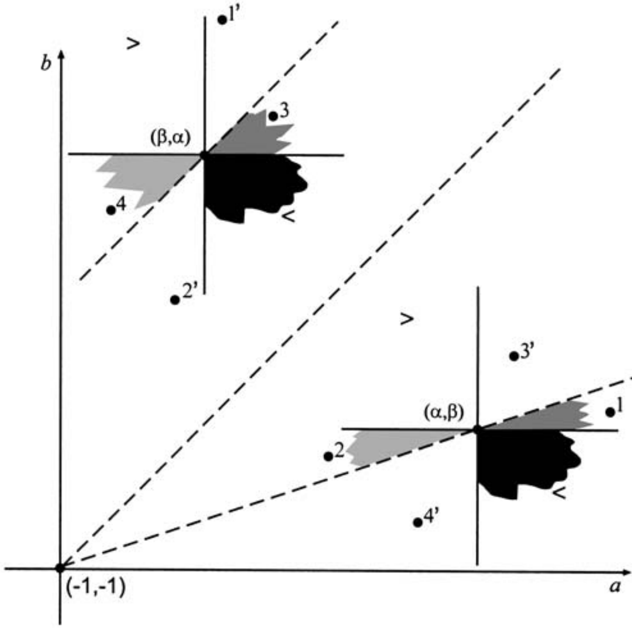

FIG. 1. Sectors in the (a, b) plane where the inequalities of Corollary 3.1 hold.

The last statement says that, when (a, b) is in the "white" regions, i.e., when (a, b) is at some of the positions indicated by primes in the figure, both the opposite inequalities $x_{n,k}(a, b) < x_{n,k}(\alpha, \beta)$ and $x_{n',k'}(a, b) > x_{n',k'}(\alpha, \beta)$ hold.

It is well known that $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(x)$. This yields

$$x_{n,k}(\alpha, \beta) = -x_{n,n+1-k}(\beta, \alpha)$$

which itself implies:

LEMMA 3.1. *Let $n \in \mathbb{N}$. Then $x_{n,k}(a, b) < x_{n,k}(\alpha, \beta)$ for every k, $1 \leqslant k \leqslant n$, if and only if $x_{n,k}(b, a) > x_{n,k}(\beta, \alpha)$ for every k, $1 \leqslant k \leqslant n$.*

Thus, the inequalities $x_{n,k}(a, b) < x_{n,k}(\alpha, \beta)$ hold for every $n \in \mathbb{N}$ and each k, $1 \leqslant k \leqslant n$ if and only if $x_{n,k}(b, a) > x_{n,k}(\beta, \alpha)$ every $n \in \mathbb{N}$ and each k, $1 \leqslant k \leqslant n$.

Proof of Theorem 1.2. Bearing in mind the statement of Corollary 3.1, all we need to prove is that, when (a, b) is in one of the "extra" sectors which contain the points 1, 2, 3, and 4 on the figure, either $x_{n,k}(a, b) < x_{n,k}(\alpha, \beta)$ or $x_{n,k}(a, b) > x_{n,k}(\alpha, \beta)$ fails. This is easily done by the last statement of Corollary 3.1 and by Lemma 3.1.

Indeed, let (a_1, b_1) be at position 1 on the figure. Let $(a_{1'}, b_{1'}) = (b_1, a_1)$. Then $(a_{1'}, b_{1'})$ is at position $1'$. On the other hand, if $x_{n,k}(a_1, b_1) < x_{n,k}(\alpha, \beta)$ for all admissible indices n and k, then, by Lemma 3.1, the inequalities $x_{n,k}(a_{1'}, b_{1'}) > x_{n,k}(\beta, \alpha)$ must also hold for all $n \in \mathbb{N}$ and k, $1 \leqslant k \leqslant n$. However, since $1'$ is in one of the white regions with respect to (β, α), by the last statement of the corollary, at least one of these inequalities must fail. We arrived at a contradiction.

Similarly, if we suppose that (a, b) is one of the positions 2, 3, or 4, and the corresponding inequalities for the zeros hold for all the pairs of indices, we shall arrive at a contradiction because the corresponding points $2'$, $3'$, or $4'$ are in white sectors.

Therefore:

1. In none of the extra sectors which contain the points 1 and 3, there exist points (a, b) for which $x_{n,k}(a, b) < x_{n,k}(\alpha, \beta)$ for all the indices n and k.
2. In none of the extra sectors which contain the points 2 and 4, there exist points (a, b) for which $x_{n,k}(a, b) > x_{n,k}(\alpha, \beta)$ for all the indices n and k.

Proof of Theorem 1.3. We consider first the even case. It follows immediately from the first formula (4.7.30) in [18] that the zeros of the polynomial

$$g_n(x; \lambda) = {}_2F_1(-n, n+\lambda; 1/2; x)$$

are $x^2_{2n,k}(\lambda)$. In other words, they coincide with the squares of the positive zeros of $C^{\lambda}_{2n}(x)$. Hence, for any nonvanishing function $f_{2n}(\lambda)$ the zeros of

$$\begin{aligned} G_n(x; \lambda) &= g_n(-x/f^2_{2n}(\lambda); \lambda) \\ &= \sum_{j=0}^{n} \binom{n}{j} \frac{(n+\lambda)_j}{(1/2)_j} \frac{1}{f^{2j}_{2n}(\lambda)} x^j \end{aligned}$$

are $-(f_{2n}(\lambda)\, x_{2n,k}(\lambda))^2$, $k = 1, \ldots, n$. Thus the products $f_{2n}(\lambda)\, x_{2n,k}(\lambda)$ are increasing functions of λ if and only if, for any sufficiently small positive ε, the polynomials $G_n(x; \lambda)$ and $G_n(x; \lambda+\varepsilon)$ form a positive pair. This is evidently equivalent to the fact that $\tilde{G}_n(x; \lambda+\varepsilon)$ and $\tilde{G}_n(x; \lambda)$ form a positive pair, where $\tilde{G}_n(x; \lambda)$ denotes the reciprocal of $G_n(x; \lambda)$. Let $\tilde{H}(C_{2n}; \lambda, \varepsilon)$ be the Hurwitz matrix associated with the polynomial

$$\tilde{G}_n(x^2; \lambda+\varepsilon) + x\tilde{G}_n(x^2; \lambda).$$

We have

$$\tilde{H}(C_{2n};\lambda,\varepsilon)=\begin{pmatrix} 1 & \binom{n}{1}\frac{(n+\lambda+\varepsilon)_1}{(1/2)_1\, f_{2n}^2(\lambda+\varepsilon)} & \binom{n}{2}\frac{(n+\lambda+\varepsilon)_2}{(1/2)_2\, f_{2n}^4(\lambda+\varepsilon)} & \cdots & 0 \\ 1 & \binom{n}{1}\frac{(n+\lambda)_1}{(1/2)_1\, f_{2n}^2(\lambda)} & \binom{n}{2}\frac{(n+\lambda)_2}{(1/2)_2\, f_{2n}^4(\lambda)} & \cdots & 0 \\ 0 & 1 & \binom{n}{1}\frac{(n+\lambda+\varepsilon)_1}{(1/2)_1\, f_{2n}^2(\lambda+\varepsilon)} & \cdots & 0 \\ 0 & 1 & \binom{n}{1}\frac{(n+\lambda)_1}{(1/2)_1\, f_{2n}^2(\lambda)} & \cdots & 0 \\ . & . & . & \cdots & . \end{pmatrix}.$$

Theorem 2.1(ii) implies that the products $f_{2n}(\lambda)\, x_{2n,k}(\lambda)$, $k=1, ..., n$, are increasing functions of λ if and only if all the principal minors $\tilde{\Delta}_j(C_{2n};\lambda,\varepsilon)$, $j=1, ..., 2n+1$, of $\tilde{H}(C_{2n};\lambda,\varepsilon)$ are positive for every sufficiently small positive ε. Observe that $\tilde{\Delta}_2(C_{2n};\lambda,\varepsilon)$ is positive if and only if

$$\frac{1}{f_{2n}^2(\lambda)}\,\frac{f_{2n}^2(\lambda+\varepsilon)-f_{2n}^2(\lambda)}{\varepsilon}>\frac{1}{\lambda+n}.$$

Letting ε tend to zero and having in mind that $f_{2n}(\lambda)$ is positive we conclude that this inequality is equivalent to (5).

Let us restrict our further considerations to the functions of the form $f_{2n}(\lambda)=(\lambda+c_{2n})^{1/2}$; then (5) yields $c_{2n}<n$. Thus $f_{2n}^2(\lambda)=\lambda+n+d$ with $d=d_{2n}<0$. The Gauss elimination process yields

$$\begin{aligned}\tilde{\Delta}_3(C_{2n};\lambda,\varepsilon)&=\frac{(n-1)(n+\lambda+\varepsilon)(n+1+\lambda+\varepsilon)}{3(n+\lambda+\varepsilon+d)^2}\\ &\quad+\frac{2n(n+\lambda)(n+\lambda+\varepsilon)}{(n+\lambda+d)(n+\lambda+\varepsilon+d)}\\ &\quad-\frac{(n-1)(n+\lambda)(n+1+\lambda)}{3(n+\lambda+d)^2}\\ &\quad-\frac{2n(n+\lambda+\varepsilon)^2}{(n+\lambda+\varepsilon+d)^2}\\ &=\frac{-\varepsilon\{A_e(n,d)\,\lambda^2+B_e(n,d)\,\lambda+C_e(n,d)\}}{3(n+\lambda+d)^2\,(n+\lambda+\varepsilon+d)^2}.\end{aligned}$$

Since the latter denominator is positive and ε is positive, then $\tilde{\Delta}_3(C_{2n}; \lambda, \varepsilon)$ is positive for large values of λ only when the leading coefficient $A_e(n, d) = 2d(2n+1)+n-1$ of the above binomial is negative. This yields (7).

The proofs of the statements concerning $f_{2n+1}(\lambda)$ are done in a similar way. Since, by the second formula (4.7.10) in [18], the zeros of the polynomial

$$h_n(x; \lambda) = {}_2F_1(-n, n+\lambda+1; 3/2; x)$$

are $x^2_{2n+1, k}(\lambda)$, then the zeros of

$$\begin{aligned} H_n(x; \lambda) &= h_n(-x/f^2_{2n+1}(\lambda); \lambda) \\ &= \sum_{j=0}^{n} \binom{n}{j} \frac{(n+\lambda+1)_j}{(3/2)_j} \frac{1}{f^{2j}_{2n+1}(\lambda)} x^j \end{aligned}$$

are $-(f_{2n+1}(\lambda)\, x_{2n+1, k}(\lambda))^2$, $k = 1, ..., n$. Therefore the products $f_{2n+1}(\lambda)$ $x_{2n+1, k}(\lambda)$ are increasing functions of λ if and only if all the principal minors $\tilde{\Delta}_j(C_{2n+1}; \lambda, \varepsilon)$ of

$$\tilde{H}(C_{2n+1}; \lambda, \varepsilon)$$

$$= \begin{pmatrix} 1 & \binom{n}{1} \frac{(n+\lambda+\varepsilon+1)_1}{(3/2)_1\, f^2_{2n+1}(\lambda+\varepsilon)} & \binom{n}{2} \frac{(n+\lambda+\varepsilon+1)_2}{(3/2)_2\, f^4_{2n+1}(\lambda+\varepsilon)} & \cdots & 0 \\ 1 & \binom{n}{1} \frac{(n+\lambda+1)_1}{(3/2)_1\, f^2_{2n+1}(\lambda)} & \binom{n}{2} \frac{(n+\lambda+1)_2}{(3/2)_2\, f^4_{2n+1}(\lambda)} & \cdots & 0 \\ 0 & 1 & \binom{n}{1} \frac{(n+\lambda+\varepsilon+1)_1}{(1/2)_1\, f^2_{2n+1}(\lambda+\varepsilon)} & \cdots & 0 \\ 0 & 1 & \binom{n}{1} \frac{(n+\lambda+1)_1}{(3/2)_1\, f^2_{2n+1}(\lambda)} & \cdots & 0 \\ . & . & . & \cdots & . \end{pmatrix}$$

are positive for any sufficiently small positive ε. Then $\tilde{\Delta}_2(C_{2n+1}; \lambda, \varepsilon)$ is positive if and only if

$$\frac{1}{f^2_{2n+1}(\lambda)} \frac{f^2_{2n+1}(\lambda+\varepsilon) - f^2_{2n+1}(\lambda)}{\varepsilon} > \frac{1}{\lambda+n+1},$$

and this is equivalent to (6). In what follows we concentrate on multipliers of the form $f_{2n+1}(\lambda) = (\lambda + c_{2n+1})^{1/2}$. In this case (6) yields $c_{2n+1} < n+1$. Let $c_{2n+1} < n+d$ with $d = d_{2n+1} < 1$. Again lengthy but straightforward calculations show that

$$\tilde{\Delta}_3(C_{2n+1}; \lambda, \varepsilon) = \frac{-\varepsilon\{A_o(n, d)\, \lambda^2 + B_o(n, d)\, \lambda + C_o(n, d)\}}{15(n+\lambda+d)^2\, (n+\lambda+\varepsilon+d)^2},$$

where $A_o(n, d) = 2(2n+3)\,d - (n+9)$. As in the previous case we conclude that $\tilde{\Delta}_3(C_{2n+1}; \lambda, \varepsilon)$ can be positive for large values of λ only when $A_o(n, d) < 0$ and this implies (8).

REFERENCES

1. S. Ahmed, M. E. Muldoon, and R. Spigler, Inequalities and numerical bound for zeros of ultraspherical polynomials, *SIAM J. Math. Anal.* **17** (1986), 1000–1007.
2. T. S. Chihara, "An Introduction to Orthogonal Polynomials," Gordon and Breach, New York, 1978.
3. D. K. Dimitrov, On a conjecture concerning monotonicity of zeros of ultraspherical polynomials, *J. Approx. Theory* **55** (1996), 88–97.
4. D. K. Dimitrov, Connection coefficients and zeros of orthogonal polynomials, *J. Comput. Appl. Math.* **133** (2001), 331–340.
5. A. Elbert and M. E. Muldoon, On the derivative with respect to a parameter of a zero of a Sturm–Liouville function, *SIAM J. Math. Anal.* **25** (1994), 354–364.
6. A. Elbert and P. D. Siafarikas, Monotonicity properties of the zeros of ultraspherical polynomials, *J. Approx. Theory* **97** (1999), 31–39.
7. F. R. Gantmacher, "The Theory of Matrices," Vol. 2, Chelsea, New York, 1959.
8. E. K. Ifantis and P. D. Siafarikas, Differential inequalities on the greatest zero of Laguerre and ultraspherical polynomials, *in* "Actas del VI Simposium on Polinomios Ortogonales Y Aplicaciones, Gijon, 1989," pp. 187–197.
9. M. E. H. Ismail, The variation of zeros of certain orthogonal polynomials, *Adv. Appl. Math.* **8** (1987), 111–118.
10. M. E. H. Ismail, Monotonicity of zeros of orthogonal polynomials, *in* "q-Series and Partitions" (D. Stanton, Ed.), pp. 177–190, Springer-Verlag, New York, 1989.
11. M. E. H. Ismail and J. Letessier, Monotonicity of zeros of ultraspherical polynomials, *in* "Orthogonal Polynomials and Their Applications" (M. Alfaro, J. S. Dehesa, F. J. Marcellan, J. L. Rubio de Francia, and J. Vinuesa, Eds.), Lecture Notes in Mathematics, Vol. 1329, pp. 329–330, Springer-Verlag, Berlin, 1988.
12. M. E. H. Ismail and M. E. Muldoon, A discrete approach to monotonicity of zeros of orthogonal polynomials, *Trans. Amer. Math. Soc.* **323** (1991), 65–78.
13. A. Laforgia, A monotonic property for the zeros of ultraspherical polynomials, *Proc. Amer. Math. Soc.* **83** (1981), 757–758.
14. M. Marden, "Geometry of Polynomials," Amer. Math. Soc. Surveys, no. 3, Providence, RI, 1966.
15. A. Markov, Sur les racines de certaines équations (second note), *Math. Ann.* **27** (1886), 177–182.
16. N. Obrechkoff, "Verteilung und Berechnung der Nullstellen reeler Polynome," VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.
17. T. J. Stieltjes, Sur les racines de l'équation $X_n = 0$, *Acta Math.* **9** (1886), 385–400.
18. G. Szegő, "Orthogonal Polynomials," 4th ed., Amer. Math. Soc. Coll. Publ., Vol. 23, Providence, RI, 1975.